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Tiling Finite Planes

Jalal Khairabadi[1], *Bahram Sadeghi Bigham*[2], *Rebvar Hosseini*[3]
Department of Computer Science, Institute for Advanced Studies in Basic Sciences[1,2,3]
Zanjan, Iran[1,2,3]

j.khair@iasbs.ac.ir, b _sadeghi_b@iasbs.ac.ir, r.hosseini@iasbs.ac.ir

Abstract: - A set of natural numbers tiles the plane if a square-tiling of the plane exists using exactly one square of side length n for every n in the set. From [2] we know that **N** itself tiles the plane. From that and [3] we know that the set of even numbers tiles the plane while the set of odd numbers does not. According to [1] it is possible to tile the plane using only an odd square. In this paper we will check that if it is possible to tile the plane using an even set and n odd numbers or not.

Keywords: Tiling; Plane; Finite Set; Finite Plane; Fibonacci

1. Introduction

In 1903, M. Behn [4] asked: Is it possible to tile a square with smaller squares, such that no square is the same size? In 1925, Moron found several rectangles that can be tiled with squares[9]. Dehn's question was answered by R. Sprague's [10] confirmation. The question and its answer was the disputable subject of a paper, \squaring an square" by Tutte [11], and was reprinted in Scientific American by Martin Gardner [12]. Several papers have been presented in this subject ever since [5-7].

In 1975, Golomb [8] asked if it is possible to tile an infinite plane using different squares with every side-length represented. In 1907, Karl Scherer [13] was assisted in tiling the plane using squares with integral sides, but in different sizes. The number of squares used with n side, t(n), is finite, but the function t is not limited. Golomb's question was answered confirmatively in \squaring the square" [2]. The solution presented caused plenty of questions, for example, which sets can tile the plane? Is it possible to tile without squared rectangles? Is there three-colored tiling? Is it possible to tile a half-plane? The second paper [3] showed that no set of odd numbers and primes can tile the plane. This paper found a kind of tiling without squared rectangles. It showed that the set of natural numbers can tile many even infinite pages. But it caused many questions. Can a superset of a tiling set tile the plane? Is it possible to partition N into two tiling and non-tiling sets? Is it possible to tile Riemann Plane? There are some relationships between squared planes and squared squares. Thus, there are some strange disconnects. It can be proved that it is not possible to cube a cube[11]. But it has not been presented that it is not possible to cube the plane.

2. Tiling Finite Planes

In this section, we will review tiling for various scenarios. Suppose that a set of even numbers is presented.

In this section, the purpose is to tile a finite plane with this set of even numbers and 1 or 2 odd numbers.

2.1. Set With One or Two Odd Number

2.1.1. Proposition 1

Definition: A set containing just one odd number can tile the plane.

Proof: Here the even numbers set is empty. Consider a square with an odd side length. Another square with the same side length can tile this square. For example a square with side length 5 can be tiled with another square with the same side length. By [1], we know that a set containing just two odd numbers and a set of evens cannot tile an infinite plane, so we have the following proposition:

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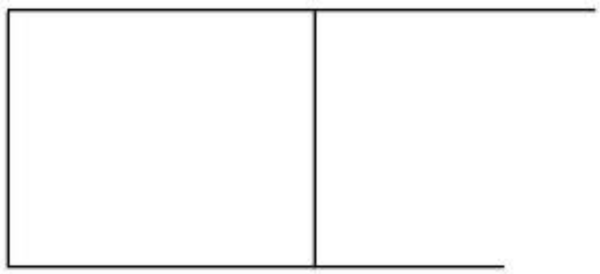

Figure 1: Integral Side

2.1.2. Proposition 2

Definition: A set containing exactly two odd numbers and a set of evens cannot tile a finite plane. To prove this proposition the following definition is needed.

Definition: At every corner of a square S there is an edge extending away from S. We will call such a line a spoke of S. We will say that S has an integral side, if it has two spokes extending in parallel from adjacent corners. If S has no integral side we say it is a Pin-wheel (Figure 1).

Proof: The finite plane is in two different states from one point of view:

1. At least, one of the sides is even.
2. Or, both sides are odd.

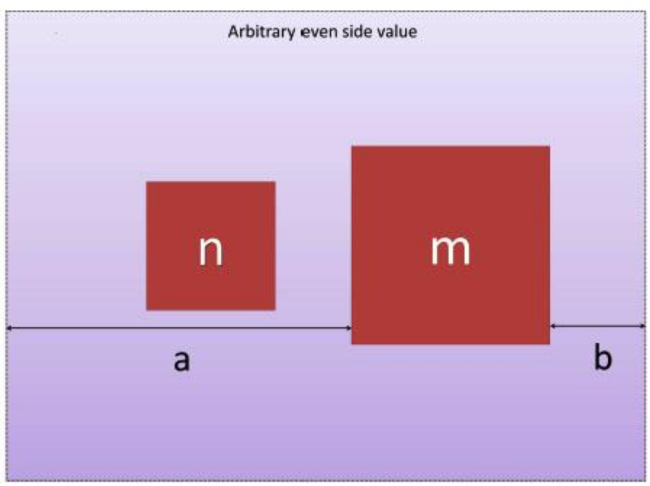

Figure 2: Plane with 2 even-value tile

Proof 1 : What if the plane has one even side? As it is shown in Figure 2, if the squares m and n with odd sides are set to the plane, one of the distances a or b is odd and since we have used both odd numbers, so it is possible to tile an odd plane with an even set.

Proof 2 : what if both sides in the finite plane are odd? The area will be odd, but if we calculate the area for an even set with two odd numbers, it will be even. So, an even set with two odd numbers cannot tile the plane.

2.2. Set With Three Odd Numbers

If we have a set of numbers including three odd numbers, this set of numbers cannot be tiled.

Proof: Here, according to the theorem, we have a set of even numbers and three odd numbers. Based on the plane that should be covered, the area of this plane is odd. Since the sum of squares for a set of an even number and three odd numbers is odd, so the area of initial plane will be odd (the length and width are odd). For these three squares with odd sides, there are many cases of how these squares are placed:

1. Two of these squares are in the same direction; which it is not possible to tile in this case, Since the sum of squares for two odd numbers and a set of even numbers will be even while the sides of tiled plane are odd.

2. All three odd squares are in the same direction Figure 3. In this case, as it is impossible to use squares in the same size according to the tiling problem, and since a part of figure will remain which is odd (a), so it is also not possible to tile the plane.

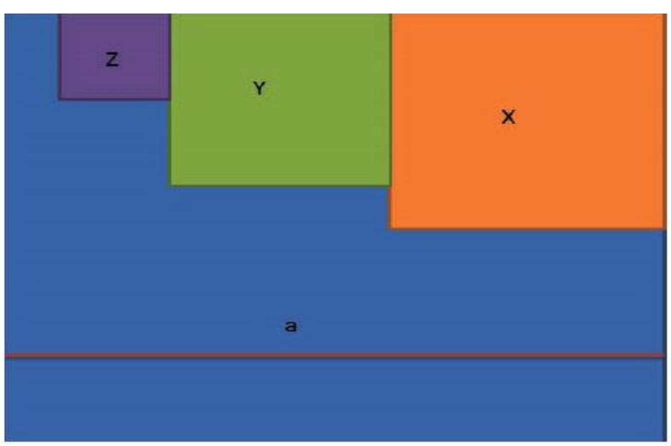

Figure 3: A plane with odd length and width

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3. Suppose one of odd squares tiles a corner completely. In this case, since one part of the plane remains (b) and its length and width are odd and even, respectively, and as proved in Theorem 2 that it is not possible to tile a plane with a set of two odd numbers, so this plane cannot be tiled, too.

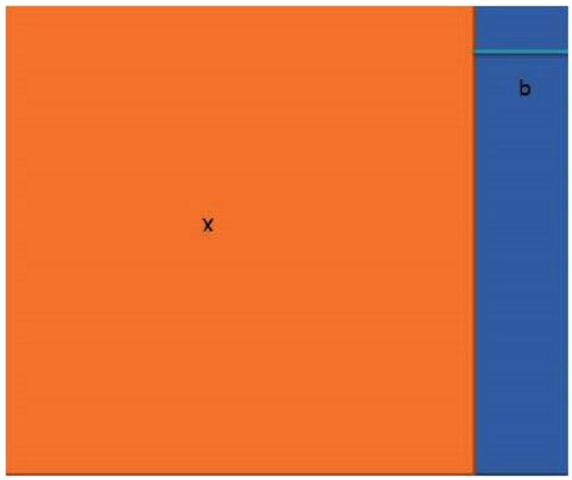

Figure 4: A plane with odd length and width

4. No square is in the same direction. In this case, since the space (a) remains so it is not possible to tile the plane. numbers, so this plane cannot be tiled, too.

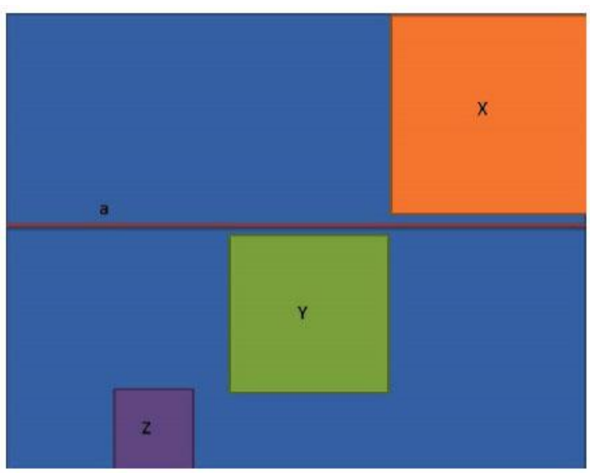

Figure 5: A plane with odd length and width

4.1 No square is in the in the same direction and all three squares are in the same direction. Since (a) is odd so it is not possible to tile the plane.

Finally, it can be proved that it is not possible to tile the plane with a set of three odd numbers.

2.3. Set With Four or Five Odd Numbers

Till now tiling finite planes with 1, 2 and 3 odd numbers and a set of evens has been covered. The next propositions will cover tiling for n ≥ 4 odds and set evens.

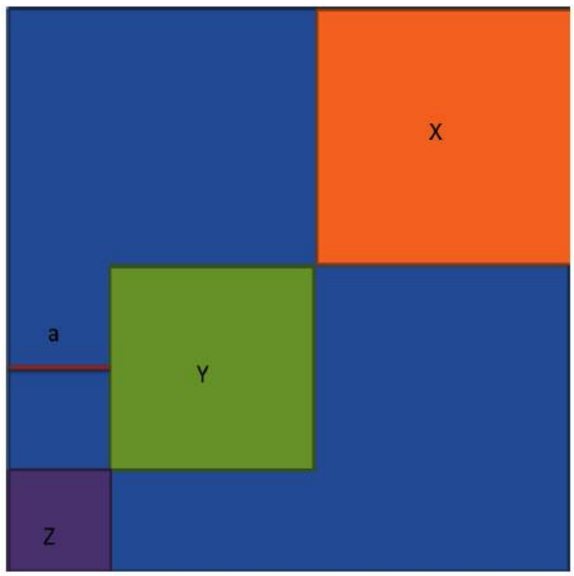

Figure 6: A plane with odd length and width

2.3.1 Proposition

Definition: It is possible to tile a finite plane by 4 odd numbers and a set of evens.

Proof: Start with a 32 * 33 squared rectangle composed of nine squares of sides 1,4,7,8,9,10,14,15,18.

This sequence of numbers can tile this finite plane. So the proposition is true.

For n = 5 another square with odd side length is needed. If we add 33 to the sequence above, there will be a square tiling for a 33 * 65 squared rectangle that contains 5 odd numbers and a set of evens.

2.4. Set With 6 Odd Numbers

Till now tiling finite planes with 1, 2 and 3 odd numbers and a set of evens has been covered. The next propositions will cover tiling for n > 4 odds and a set of evens.

2.4.1 Proposition

Definition: It is possible to tile a finite plane by 6 odd numbers and a set of evens.

Proof: Start with a 98 * 65 squared rectangle composed of nine squares of sides 1, 4, 7, 8, 9, 10, 14, 15, 18, 33, 65. This sequence of numbers can tile this finite plane. So the proposition is true.

2.5. Set With n ≥ 7 Odd Numbers

Theorem 1: For n ≥ 7 the sequence 1; 4; 7; 8; 9; 10; 14; 15; 18; 33; 65 and then Fibonacci rule sequence starting from 88 can tile a finite plane with n ≥ 7 odd numbers and a set of evens.

Proof: Till now, tiling for different values of 1 ≤ n ≤ 6 with the following sequences is possible:

n= 4: 1,4,7,8,9,10,14,15,18

n= 5 : 1,4,7,8,9,10,14,15,18,33

n= 6 : 1,4,7,8,9,10,14,15,18,33,65

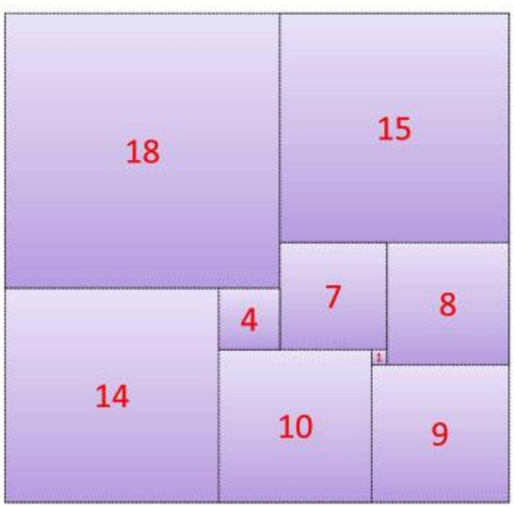

Figure 7: Tiling Example, 32×33

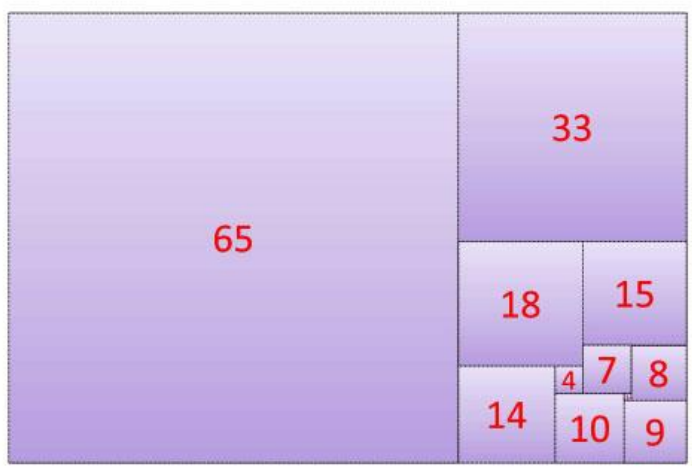

Figure 8: Tiling Example, 33×65

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From this point on, adding two last numbers in this sequence will give us the next number like the Fibonacci sequence. If next number is odd, then the sequence for the next n is complete and tiling with this sequence is possible. Otherwise, adding two last numbers again, will give us the next number, this number is odd. So the sequence is complete. There is a axiom here: if the last two numbers k; l in the sequence are odd, then the following two numbers are required: k + l and l + k + l This is just like the Fibonacci sequence rule. Otherwise, if one of k and l is odd and the other is even adding just k + l to the sequence is sufficient. Also, two even numbers except 10, 14 are impossible. So the proof is complete. Note: if a set grows faster than Fibonacci sequence, it is not possible to tile plane with it [1].

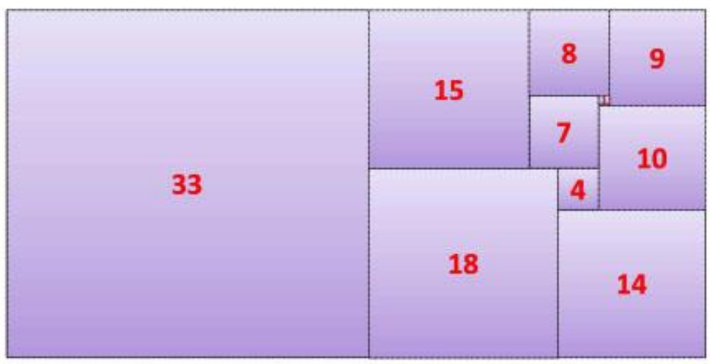

Figure 9: Tiling Example, 98×65

3. Conclusion

In this paper, we focused on the sets including n odd numbers. It was proved that it was not possible to tile the plane with a set of even numbers and **n = 2; 3** odd numbers. It was also proved that it was not possible to tile the plane with a set of even numbers and **n** odd numbers (**n** $\neq$ 2; 3).

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Authors profile:

Jalal Khairabadi received his BS degree from the Kermanshah Jahad Daneshgahi university in computer engineering in 2010 and the MS degree from Institute for Advanced Studies in Basic Sciences in Computer Science in 2013. His research interests are Algorithms, Computational Geometry and Multi-Objective Optimization. He is a member of RoboCG-lab, DM-lab and MAS-lab in the institute for Advanced Studies in Basic Sciences.

